Self-Diffusion by Multivariate-Normal Turbulent Velocity Field

R. V. R. Pandya*,
Department of Mechanical Engineering,
University of Puerto Rico at Mayaguez, Mayaguez, PR 00680, USA

February 2, 2008

Abstract

A closed set of *exact* equations describing statistical theory of turbulent self-diffusion by multivariate-normal turbulent velocity field is derived. In doing so, we first suggest exact formulas for correlations $\langle f_i(p) f_j(p') R[\mathbf{f}]\rangle$, $\langle g(p) R[\mathbf{f}]\rangle$ and $\langle g(p) f_j(p') R[\mathbf{f}]\rangle$ when the functional $R[\mathbf{f}]$ is functional of functions f_i's having multivariate-normal distribution, g and f_i's have joint normal distribution and zero mean values.

Originative works of Taylor [1] and Batchelor [2] on turbulent self-diffusion of fluid particles in Lagrangian and Eulerian frameworks, respectively, have placed statistics of Lagrangian velocity field, first put forward by Taylor [1], as fundamental properties in the field of statistical theory of diffusion. Despite many persistent efforts (for review, see e.g., Refs. [3, 4]), the theory of diffusion remains incomplete mainly due to involved closure problems [4, 5, 6] and lack of accurate prediction of Lagrangian statistics [7, 8, 9] in general turbulent flows. In this letter, we base our Eulerian analysis on the equation for a passive scalar field $\psi(\mathbf{x}, t)$ whose ensemble average (denoted by $\langle\,\rangle$) denotes Green's function G for the evolution in space-time $(\mathbf{x} - t)$ of an arbitrary initial probability distribution for the position of marked fluid particles in space [2, 5, 8]. We solve the closure problem arising in the equation for $\langle\psi(\mathbf{x}, t)\rangle$ resulting in an appearance of Lagrangian velocity correlations. We then obtain equations relating Lagrangian and Eulerian correlations for the velocity field. These equations are exact when the fluctuations u'_i in turbulent velocity field $u_i(\mathbf{x}, t) = U_i(\mathbf{x}, t) + u'_i(\mathbf{x}, t)$ over the mean velocity $U_i(\mathbf{x}, t)$ obey multivariate-normal distribution. In doing so, we first suggest three new functional formulas which along with the Furutsu-Donsker-Novikov functional formula [10] are used in our Eulerian analysis. These four functional formulas are now discussed briefly before presenting the Eulerian analysis.

*Email address: *rvrpturb@uprm.edu*

Functional Formulas: Furutsu-Donsker-Novikov formula [10] suggests that for any random function $f_i(p)$ with normal or Gaussian distribution and $\langle f_i(p)\rangle = 0$, its correlation with functional $R[\mathbf{f}]$ of function f_i's can be exactly given by

$$\langle f_i(p)R[\mathbf{f}]\rangle = \int \langle f_i(p)f_k(p')\rangle \left\langle \frac{\delta R[\mathbf{f}]}{\delta f_k(p')dp'} \right\rangle dp', \tag{1}$$

where the integral extends over the region of arguments p in which the function f_i is defined and $\left\langle \frac{\delta R[\mathbf{f}]}{\delta f_k(p')dp'} \right\rangle$ represents the functional derivative of R with respect to f_k. Here, we suggest another exact functional formula for higher order correlations $\langle f_i(p)f_j(p')R[\mathbf{f}]\rangle$, written as

$$\begin{aligned}\langle f_i(p)f_j(p')R[\mathbf{f}]\rangle = \quad & \langle f_i(p)f_j(p')\rangle\langle R[\mathbf{f}]\rangle + \frac{1}{2}\int \langle f_i(p)f_k(s)\rangle \left\langle f_j(p')\frac{\delta R[\mathbf{f}]}{\delta f_k(s)ds} \right\rangle ds \\ & + \frac{1}{2}\int \langle f_j(p')f_k(s)\rangle \left\langle f_i(p)\frac{\delta R[\mathbf{f}]}{\delta f_k(s)ds} \right\rangle ds. \end{aligned} \tag{2}$$

Yet another formulas which can be obtained from Eqs. (1) and (2), respectively, are

$$\langle g(p)R[\mathbf{f}]\rangle = \int \langle g(p)f_k(p')\rangle \left\langle \frac{\delta R[\mathbf{f}]}{\delta f_k(p')dp'} \right\rangle dp', \tag{3}$$

$$\begin{aligned}\langle g(p)f_j(p')R[\mathbf{f}]\rangle = \quad & \langle g(p)f_j(p')\rangle\langle R[\mathbf{f}]\rangle + \frac{1}{2}\int \langle g(p)f_k(s)\rangle \left\langle f_j(p')\frac{\delta R[\mathbf{f}]}{\delta f_k(s)ds} \right\rangle ds \\ & + \frac{1}{2}\int \langle f_j(p')f_k(s)\rangle \left\langle g(p)\frac{\delta R[\mathbf{f}]}{\delta f_k(s)ds} \right\rangle ds \end{aligned} \tag{4}$$

where g and f_i's have joint normal distribution and $\langle g\rangle = 0$. It should be noted that the terms with functional derivatives in Eqs. (2) and (4) can be further simplified by using Eqs. (1) and (3). From now onward, these functional formulas given by Eqs. (1), (2), (3) and (4) are referred to as $F1$, $F2$, $F3$ and $F4$, respectively.

Now, we briefly describe equations which can be used for verifications of the functional formulas. Using Novikov's notation, expansion of $R[\mathbf{f}]$ around $\mathbf{f} = \mathbf{0}$ can be written as

$$R[\mathbf{f}] = R[\mathbf{0}] + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int R^{(n)}_{i_1\cdots i_n}(s_1, \cdot, s_n) f_{i_1}(s_1)\cdots f_{i_n}(s_n) ds_1 \cdots ds_n \tag{5}$$

where

$$R^{(n)}_{i_1\cdots i_n}(s_1, \cdot, s_n) = \frac{\delta^n R[\mathbf{f}]}{\delta f_{i_1}(s_1)ds_1 \cdots \delta f_{i_n}(s_n)ds_n}\Big|_{\mathbf{f}=\mathbf{0}} \tag{6}$$

and summation over the repeated indices $i_1, i_2 \cdots i_n$ is taken. The functional derivative of Eq. (5) yields

$$\frac{\delta R[\mathbf{f}]}{\delta f_k(s)ds} = \frac{\delta R[\mathbf{f}]}{\delta f_k(s)ds}\Big|_{\mathbf{f=0}} + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int \cdots \int R^{(n)}_{ki_2\cdots i_n}(s, s_2, \cdots, s_n) f_{i_2}(s_2) \cdots f_{i_n}(s_n) ds_2 \cdots ds_n. \tag{7}$$

For multivariate-normal distribution for f_i's and joint normal distribution for g and f_i's

$$\langle A(s) f_{i_1}(s_1) \cdots f_{i_m}(s_m)\rangle = 0, \tag{8}$$

$$\langle A(s) f_{i_1}(s_1) \cdots f_{i_n}(s_n)\rangle = \sum_{\alpha=1}^{n} \langle A(s) f_{i_\alpha}(s_\alpha)\rangle \langle f_{i_1}(s_1) \cdots f_{i_{\alpha-1}}(s_{\alpha-1}) f_{i_{\alpha+1}}(s_{\alpha+1}) \cdots f_{i_n}(s_n)\rangle \tag{9}$$

where $A(s)$ is equal to any $f_i(s)$ or $g(s)$ and m is an even integer. Using Eqs. (5)-(9), formulas $F2$, $F3$ and $F4$ can be verified.

Eulerian Analysis: We consider scalar field

$$\psi(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{X}(\mathbf{t})) \tag{10}$$

for the Eulerian description of the self-diffusion of fluid particle whose position $\mathbf{X}(t)$ (also denoted by components $X_i(t)$) at any time t is governed by the Lagrangian equation

$$\frac{dX_i(t)}{dt} = u_i(\mathbf{x} = \mathbf{X}(t), t) \tag{11}$$

with initial condition $\mathbf{X}(t_0) = \mathbf{x}_0$ at time t_0. The governing equation for ψ can be written as (see e.g. [5])

$$\frac{\partial}{\partial t}\psi(\mathbf{x}, t) + \frac{\partial}{\partial x_i}[\psi(\mathbf{x}, t) u_i(\mathbf{x}, t)] = 0, \ \forall\, t > t_0 \tag{12}$$

with initial condition $\psi(\mathbf{x}, t_0) = \delta(\mathbf{x} - \mathbf{x_0})$. The ensemble average of ψ multiplied by $d\mathbf{x}$, i.e. $\langle\psi\rangle d\mathbf{x} \equiv G(\mathbf{x}, \mathbf{t}|\mathbf{x}_0, t_0) d\mathbf{x}$, with the Green's function G, denotes the probability that a fluid particle at $(\mathbf{x}_0, t_0)$ should lie within a volume $d\mathbf{x}$ at later time $t > t_0$ [5]. The ensemble average of Eq. (12)

$$\frac{\partial}{\partial t}\langle\psi(\mathbf{x}, t)\rangle + \frac{\partial}{\partial x_i}[\langle\psi\rangle U_i(\mathbf{x}, t) + \langle\psi u'_i(\mathbf{x}, t)\rangle] = 0 \tag{13}$$

with initial condition $\langle\psi(\mathbf{x}, t_0)\rangle = \delta(\mathbf{x} - \mathbf{x_0})$ poses closure problem due to unknown correlation $\langle\psi(\mathbf{x}, t) u'_i(\mathbf{x}, t)\rangle$. In our analysis, we would need yet another scalar function $\psi_b(\mathbf{y}, t-s) = \delta(\mathbf{y} - \mathbf{X}(t-s))$ with initial condition given at time t i.e. $\psi_b(\mathbf{y}, t) = \langle\psi_b(\mathbf{y}, t)\rangle = \delta(\mathbf{y} - \mathbf{X}(t)) = \delta(\mathbf{y} - \mathbf{x})$. In fact, $\langle\psi_b(\mathbf{y}, t-s)\rangle d\mathbf{y}$ with $s > 0$ represents the probability that a fluid particle lies within a volume $d\mathbf{y}$ at $(\mathbf{y}, t-s)$ if the particle reaches $(\mathbf{x}, t)$. The governing equation for $\psi_b(\mathbf{y}, t-s)$, after introducing notation $s^- \equiv t - s$, is

$$\frac{\partial}{\partial s}\psi_b(\mathbf{y}, s^-) - \frac{\partial}{\partial y_i}[\psi_b(\mathbf{y}, s^-) u_i(\mathbf{y}, s^-)] = 0, \tag{14}$$

which is obtained by using the Lagrangian equation

$$-\frac{d}{ds}X_i(s^-) = u_i(\mathbf{y} = \mathbf{X}(s^-), s^-), \forall\, 0 \le s \le t - t_0 \tag{15}$$

describing the backward evolution of trajectory of fluid particle with known initial position $\mathbf{X}(t)$ at initial time $s^- = t$ or $s = 0$. The ensemble average of Eq. (14) poses closure problem due to the unknown correlation $\langle \psi_b(\mathbf{y}, s^-) u_i'(\mathbf{y}, s^-) \rangle$.

Now we obtain exact closed expressions for the unknown correlations by using $F1$ and following a method similar to that provided earlier in the context of kinetic approach for two-phase turbulent flows[11]. Considering $\psi[\mathbf{u}']$ as a functional of u_i''s having multivariate-normal distribution and applying $F1$, we obtain

$$\langle \psi(\mathbf{x}, t) u_i'(\mathbf{x}, t) \rangle = -\frac{\partial}{\partial x_k}[\lambda_{ki} \langle \psi \rangle] + \gamma_i \langle \psi \rangle \tag{16}$$

where tensors

$$\lambda_{ki} = \int_{t_0}^{t} \langle u_i'(\mathbf{x}, t) u_j'(t|t_2) \rangle G_{jk}(t_2|t) dt_2, \tag{17}$$

$$\gamma_i = \int_{t_0}^{t} \left\langle \frac{\partial u_i'(\mathbf{x}, t)}{\partial x_r} u_j'(t|t_2) \right\rangle G_{jr}(t_2|t) dt_2, \tag{18}$$

and shorthand notation $u_i'(t|t_2)$ is used to represent u_i' at $(\mathbf{X}(t_2), t_2)$ along the fluid particle path which reaches $\mathbf{x}$ at t. Also G_{jk} is governed by

$$\frac{d}{dt}G_{jk}(t_2|t) - \frac{\partial U_k(\mathbf{x}, t)}{\partial x_a} G_{ja}(t_2|t) = \delta_{jk}\delta(t - t_2) \tag{19}$$

and $G_{jk} = \delta_{jk}$ for homogeneous turbulence with uniform mean velocity U_k. The λ_{ki} represents eddy diffusivity and $\gamma_i - \frac{\partial \lambda_{ki}}{\partial x_k}$ represents drift velocity when Eq. (16) is substituted in Eq. (13). We should mention that for the homogeneous turbulence with constant U_k, the present solution for $\langle \psi u_i' \rangle$ becomes identical to the solution derived by Reeks [12]. Also, the present solution is valid for both incompressible and compressible velocity fields. Similarly, considering ψ_b as a functional of $u_i'(\mathbf{y}, s^-)$ and applying $F1$, we obtain

$$\langle \psi_b(\mathbf{y}, s^-) u_i'(\mathbf{y}, s^-) \rangle = -\frac{\partial}{\partial y_k}[\lambda_{ki}^b \langle \psi_b \rangle] + \gamma_i^b \langle \psi_b \rangle \tag{20}$$

where the tensors

$$\lambda_{ki}^b = \int_0^s \langle u_i'(\mathbf{y}, s^-) u_j'(s^-|s_2^-) \rangle G_{jk}^b(s_2^-|s^-) ds_2, \tag{21}$$

$$\gamma_i^b = \int_0^s \left\langle \frac{\partial u_i'(\mathbf{y}, s^-)}{\partial y_r} u_j'(s^-|s_2^-) \right\rangle G_{jr}^b(s_2^-|s^-) ds_2, \tag{22}$$

and $u'_j(s^-|s_2^-)$ represents u'_j at time $s_2^- \equiv t - s_2$ along the particle trajectory which passes through $\mathbf{y}$ at time $s^- \equiv t - s$. Also, $G^b_{jk}(s_2^-|s^-)$ satisfies

$$-\frac{d}{dt}G^b_{jk}(s_2^-|s^-) - \frac{\partial U_k(\mathbf{y}, s^-)}{\partial y_a}G^b_{ja}(s_2^-|s^-) = \delta_{jk}\delta(s^- - s_2^-). \qquad (23)$$

The right hand side (rhs) of Eqs. (17), (18), (21) and (22), contain unknown Lagrangian velocity field correlations $\langle u'_i(\mathbf{x}, t)u'_j(t|t_2)\rangle$, $\langle u'_j(t|t_2)\partial u'_i(\mathbf{x}, t)/\partial x_r\rangle$, $\langle u'_i(\mathbf{y}, s^-)u'_j(s^-|s_2^-)\rangle$ and $\langle u'_j(s^-|s_2^-)\partial u'_i(\mathbf{y}, s^-)/\partial y_r\rangle$, respectively. The exact equations relating these correlations to the Eulerian velocity field correlations can be derived by using the functional formulas. Using ψ and ψ_b, the correlations can be written in the forms

$$\langle a'_{[a']}u'_j(t|t_2)\rangle = \int \langle a'_{[a']}u'_j(\mathbf{y}_2, s_2^-)\psi_b(\mathbf{y}_2, s_2^-)\rangle d\mathbf{y}_2, \qquad (24)$$

$$\langle c'_{[c']}u'_j(s^-|s_2^-)\rangle = \int \langle c'_{[c']}u'_j(\mathbf{x}_2, t_2)\psi(\mathbf{x}_2, t_2)\rangle d\mathbf{x}_2, \qquad (25)$$

where $a'_{[a']} \equiv u'_i(\mathbf{x}, t)$ or $a'_{[a']} \equiv \partial u'_i(\mathbf{x}, t)/\partial x_r$ and $c'_{[c']} \equiv u'_i(\mathbf{y}, s^-)$ or $c'_{[c']} \equiv \partial u'_i(\mathbf{y}, s^-)/\partial y_r$. Here in Eqs. (24) and (25), we have introduce another notations $[a']$ and $[c']$ for subscripts and which are equivalent to the subscripts of selected variables for $a'_{[a']}$ and $c'_{[c']}$. For example, $[a'] \equiv i$ and $[a'] \equiv ir$ when $a'_{[a']} \equiv u_i$ and $a'_{[a']} \equiv \partial u_i/\partial x_r$, respectively. Also, $\psi_b(\mathbf{y}_2, s_2^-) = \delta(\mathbf{y}_2 - \mathbf{X}(s_2^-))$ and $t_2 = t - s_2 \equiv s_2^-$ with initial condition given at time t or $s = 0$ and initial condition for $\psi(\mathbf{x}, t_0) = \delta(\mathbf{x} - \mathbf{y})$ is given at time $t_0 = t - s$. It should be noted that coupling between the equations for $\langle\psi\rangle$ and $\langle\psi_b\rangle$ arises due to Eqs. (24) and (25). We now obtain expressions for third order correlations appearing on the rhs of Eqs. (24) and (25).

We introduce a few shorthand notations as $(\mathbf{a}) \equiv (\mathbf{x}, t)$, $(\mathbf{b}) \equiv (\mathbf{y}, s^-)$, $(\mathbf{n}) \equiv (\mathbf{y}_n, s_n^-)$ and $d\mathbf{n} \equiv d\mathbf{y}_n\, ds_n$ and $t_n = t - s_n \equiv s_n^-$ for $n = 1, 2, \cdots$. Now consider general form $\langle a'_{[a']}u'_j(\mathbf{y}, s^-)\psi_b(\mathbf{y}, s^-)\rangle$ with ψ_b as a functional of $u'_j(\mathbf{y}, s^-)$ and by applying $F1, F2$ or $F3, F4$ depending on whether $a'_{[a']}$ is equal to u'_i or not, we can obtain expression for the third order correlation as

$$\langle a'_{[a']}(\mathbf{a})u'_j(\mathbf{b})\psi_b(\mathbf{b})\rangle = \langle a'_{[a']}u'_j(\mathbf{b})\rangle\langle\psi_b(\mathbf{b})\rangle + \mathcal{A}_{[a']j}, \qquad (26)$$

with

$$\mathcal{A}_{[a']j} = [\frac{\partial^2}{\partial y_p \partial y_l}\Lambda_{[a']jpl} - \frac{\partial}{\partial y_p}\Omega_{[a']jp} - \frac{\partial}{\partial y_l}\Pi_{[a']jl} + \Gamma_{[a']j}]\langle\psi_b\rangle \qquad (27)$$

where the tensor

$$\Lambda_{[a']jpl} = \int_0^s ds_1 \int_0^s ds_3 \langle a'_{[a']}u'_k(s^-|s_1^-)\rangle\langle u'_j(\mathbf{b})u'_q(s^-|s_3^-)\rangle G^b_{qp}(s_3^-|s^-)G^b_{kl}(s_1^-|s^-) \qquad (28)$$

and $(s^-|s_n^-)$ is used to represent value of velocity fluctuation at $(\mathbf{X}(t-s_n), t-s_n)$ along the fluid particle trajectory which passes through $\mathbf{y}$ at time $t-s$. It should be noted that $0 \leq s_n \leq s$. The expressions for remaining tensors $\Omega_{[a']jp}$, $\Pi_{[a']jl}$ and $\Gamma_{[a']j}$ in Eq. (27) are equivalent to the rhs of Eq. (28) but with changes $u'_j(\mathbf{b}) \to \frac{\partial u'_j(\mathbf{b})}{\partial y_l}$, $u'_j(\mathbf{b}) \to \frac{\partial u'_j(\mathbf{b})}{\partial y_p}$ and $u'_j(\mathbf{b}) \to \frac{\partial^2 u'_j(\mathbf{b})}{\partial y_p \partial y_l}$, respectively. It should be noted that substituting only the first term on the rhs of Eq. (26) into Eq. (24) would yield results obtained by using Corrsin's hypothesis [7]. The remaining non-zero terms $\mathcal{A}_{[a']j}$ and $\mathcal{B}_{[a']j}$ in Eq. (26) account for the correlation between $a'_{[a']}, u'_j$ and $\psi_b - \langle\psi_b\rangle$.

Now, the tensors present in Eq. (27) contain Lagrangian correlations of the form $\langle c'_{[c']} u'_j(s^-|s_n^-)\rangle$ and for which expressions can be written similar to that of Eq. (25). To obtain closed expression for the third order correlation appearing in Eq. (25), we consider general form $\langle c'_{[c']} u'_j(\mathbf{a})\psi(\mathbf{a})\rangle$ and by applying $F1, F2$ or $F3, F4$ depending on whether $c'_{[c']}$ is equal to u'_i or not, we obtain

$$\langle c'_{[c']} u'_j(\mathbf{a})\psi(\mathbf{a})\rangle = \langle c'_{[c']} u'_j(\mathbf{a})\rangle\langle\psi(\mathbf{a})\rangle + \mathcal{C}_{[c']j} \tag{29}$$

with

$$\mathcal{C}_{[c']j} = [\frac{\partial^2}{\partial x_p \partial x_l}\lambda_{[c']jpl} - \frac{\partial}{\partial x_p}\omega_{[c']jp} - \frac{\partial}{\partial x_l}\pi_{[c']jl} + \gamma_{[c']j}]\langle\psi\rangle \tag{30}$$

where the tensor

$$\lambda_{[c']jpl} = \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_3 \langle c'_{[c']} u'_k(t|t_1)\rangle\langle u'_j(\mathbf{a}) u'_q(t|t_3)\rangle G_{qp}(t_3|t) G_{kl}(t_1|t) \tag{31}$$

The expressions for remaining tensors $\omega_{[c']jp}$, $\pi_{[c']jl}$ and $\gamma_{[c']j}$ in Eq. (30) are equivalent to the rhs of Eq. (31) but with changes $u'_j(\mathbf{a}) \to \frac{\partial u'_j(\mathbf{a})}{\partial x_l}$, $u'_j(\mathbf{a}) \to \frac{\partial u'_j(\mathbf{a})}{\partial x_p}$ and $u'_j(\mathbf{a}) \to \frac{\partial^2 u'_j(\mathbf{a})}{\partial x_p \partial x_l}$, respectively. Further, the tensors present in Eq. (30) contain Lagrangian correlations of the form $\langle a'_{[a']} u'_j(t|t_n)\rangle$, similar to Eq. (24) and for which expressions are derived above.

Concluding Remarks: In this letter, it has been shown that a closed set of exact statistical equations describing the turbulent self-diffusion by multivariate-normal velocity field can be obtained after solving involved closure problems by using Furutsu-Donsker-Novikov formula along with other three functional formulas. The only unknowns in the set are various Eulerian correlations of the velocity field. We have seen that the statistical theory of the diffusion requires two scalar fields ψ and ψ_b describing the forward and backward calculation of the diffusion with initial conditions given at time $t_0 < t$ and t, respectively, for the evolutions of ψ and ψ_b. Further it has been shown that the Lagrangian correlations appearing in the theory can be related to Eulerian correlations through the use of ψ and ψ_b and solving the involved closure problems in an exact manner by using the functional formulas. The Eulerian analysis presented in

this letter is valid for both homogeneous and inhomogeneous turbulent velocity field. Lastly, we should mention that the functional formulas can also be used to derive expressions for Lagrangian correlations of fluid velocity field along the inertial particles which occur in the kinetic approach for dispersed phase of particles/droplets in turbulent flows (e.g., see Refs. [13] and references cited therein).

References

[1] G. I. Taylor, Proc. Lond. Math. Soc. (2) **20**, 196 (1921).

[2] G. K. Batchelor, Austal. J. Sci. Res. **2**, 437 (1949).

[3] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics, Vol. 2* (MIT Press, ADDRESS, 1975); D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon Press, Oxford, 1973); J. O. Hinze, *Turbulence* (McGraw Hill Book Company, New York, NY, 1975); G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Physics **73**, 913 (2001); O. G. Bakunin, Rep. Prog. Phys. **67**, 965 (2004).

[4] W. D. McComb, *The Physics of Fluid Turbulence* (Oxford University Press, New York, NY, 1990).

[5] P. H. Roberts, J. Fluid Mechanics **11**, 257 (1961).

[6] R. H. Kraichnan, Phys. Fluids **9**, 1937 (1966).

[7] S. Corrsin, Adv. in Geophysics **6**, 161 (1959).

[8] R. H. Kraichnan, Phys. Fluids **13**, 22 (1970).

[9] P. G. Saffman, Appl. Sci. Res. **11**, 245 (1962); R. Phythian, J. Fluid Mech. **67**, 145 (1975); J. Weinstock, Phys. Fluids **19**, 1702 (1976); T. S. Lundgren and Y. B. Pointin, Phys. Fluids **19**, 355 (1976); Y. Kaneda and T. Gotoh, Phys. Fluids A **3**, 1924 (1991).

[10] K. Furutsu, J. Res. Natl. Bur. Stand. D **67**, 303 (1963); M. D. Donsker, in *Analysis in Function Space*, edited by W. A. Martin and I. Segal (MIT, Cambridge, MA, 1964); E. A. Novikov, Sov. Phys. - J. Exp. Theoret. Phys. **20**, 1290 (1965).

[11] K. E. Hyland, S. McKee, and M. W. Reeks, J. Phys. A: Math. Gen. **32**, 6169 (1999); R. V. R. Pandya and F. Mashayek, AIAA Journal **41**, 841 (2003).

[12] M. W. Reeks, *Paper 187, presented at 4th International Conference on Multiphase Flow, May 27-June 1* (PUBLISHER, New Orleans, LA, USA, 2001).

[13] R. V. R. Pandya and F. Mashayek, J. Fluid Mech. **475**, 205 (2003); R. V. R. Pandya, P. Stansell, and J. Cosgrove, Phys. Rev. E **70**, 025301(R) (2004).